THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050B Mathematical Analysis I (Fall 2016)
Tutorial for 10 Nov

We adopt the following notations: Let $A \subseteq \mathbb{R}$ be nonempty, $c \in \mathbb{R}$ be a cluster point of A, and $f, g : A \to \mathbb{R}$ be a function.

1. (a) (Limits and continuity) With notations above, f is continuous at c if and only if $\lim_{x \to c} f(x)$ exists and is equal to $f(c)$.

   (b) (Sequential Criterion for continuity) With notations above, f is continuous at c if and only if for each sequence $(x_n) \subseteq A$ converging to c, we have

   $$\lim_{n \to \infty} f(x_n) = f(c)$$

   (c) (Cauchy criterion) With notations above, define the oscillation of f at c in a $\delta > 0$ neighbourhood as:

   $$O_f^\delta(c) := \sup\{|f(x) - f(y)| : |x - c| < \delta, |y - c| < \delta\}$$

   Show that f is continuous at c if and only if

   $$\lim_{\delta \to 0^+} O_f^\delta(c) = 0$$

2. (a) (Computational Rules) Let f, g be as above, and suppose f, g are continuous at c. Then we have:

   i. $f + g$ is continuous at c.
   ii. cf is continuous at c, for any $c \in \mathbb{R}$.
   iii. fg is continuous at c.
   iv. If in addition $f(c) \neq 0$, then $\frac{1}{f}$ is continuous at c.

   (b) (Composition) Let $f : [a, b] \to [c, d]$, and $g : [c, d] \to \mathbb{R}$. Show that if f is continuous at $x_0 \in [a, b]$ and g is continuous at $f(x_0) \in [c, d]$, then $g \circ f : [a, b] \to \mathbb{R}$ is continuous at x_0. Compare with the case of limits of functions.

3. Using the definition or the properties listed above, find the set of continuity points for each of the following functions:

   (a) $f : \mathbb{R} \to \mathbb{R}$, f is a polynomial.

   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := \chi_{\mathbb{Q}}(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$

   (c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x\chi_{\mathbb{Q}}(x) := \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$

4. (Linear functions, Optional) Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ for each $x, y \in \mathbb{R}$. Further suppose there exists $x_0 \in \mathbb{R}$ at which f is continuous. Show that there exists a unique $c \in \mathbb{R}$ such that $f(x) = cx$ for any $x \in \mathbb{R}$.